Int. J. Solids Structures Vol. 30, No. 5, pp. 601–606, 1993
Printed in Great Britain

0020–7683/93 \$6.00+ .00

WARPING OF PRISMATIC BARS IN TORSION

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(*Received* 29 *February* 1992; *in revised form* 10 *September* 1992)

Abstract—St Venant's theory of torsion assumes that equal but opposite axial couples acting on the ends of a uniform elastic shaft result in every cross-section rotating about an axial line without inplane deformation. The location of this line, called the center of twist, is not specified: for small twist, it seemingly could be any line parallel to the generators of the undeformed shaft. For large twist however, clearly there is a unique center. This note examines the problem of finding the warping and center of twist of any prismatic shaft for infinitesimal deformation.

Axial torque on a prismatic shaft causes out-of-plane warping of noncircular cross-sections in addition to inplane rotation about a center of twist. If the shaft has uniform twist α per unit length and an axial coordinate $\bar{z}$ measured from one end, each cross-section has rotation $\alpha\bar{z}$. For Cartesian coordinates $\bar{x}, \bar{y}$ on axes originating at the center of twist, the cross-section has components of inplane displacement $\bar{u}(\bar{x}, \bar{y})$, $\bar{v}(\bar{x}, \bar{y})$ and axial displacement $\bar{w}(\bar{x}, \bar{y})$ that can be expressed as

$$\bar{u} = -\alpha\bar{y}\bar{z}, \quad \bar{v} = \alpha\bar{x}\bar{z}, \quad \bar{w} = \alpha\bar{\psi}(\bar{x}, \bar{y}), \tag{1}$$

where $\bar{\psi}(\bar{x}, \bar{y})$ is the warping function that gives the normal component of displacement on every cross-section. This displacement field is the starting point for St Venant's theory of torsion. For negligible body forces, these displacements give elastic stresses that are in equilibrium and tractions that vanish on lateral surfaces S if the warping function satisfies

$$\nabla^2\bar{\psi} = 0,$$
$$2\mathbf{n}\cdot\nabla\bar{\psi} = \mathrm{d}\bar{r}^2/\mathrm{d}s \ \text{ on } \ S, \tag{2}$$

where $\mathbf{n}$ is the unit vector in the outward normal direction at any point on S, $\bar{r} = \sqrt{\bar{x}^2+\bar{y}^2}$ is the transverse distance from the origin to a point, s is the arc length along the perimeter of the cross-section and ∇ is the gradient differential operator.

Frequently it is advantageous to choose a coordinate system x, y with axes originating at a point in the cross-section other than the center of twist. This is especially true if the section has less than two axes of symmetry for then the location of the center of twist may be unknown. This note examines the problem of finding the warping and center of twist of any prismatic shaft for an arbitrarily located origin of transverse coordinates. Our formulation uses Cicala's definition of the center of twist. Weinstein (1947) showed that this definition gives a center of small twist that is coincident with the local or Trefftz definition for the center of shear;† as Southwell (1941) pointed out, this coincidence is necessary to satisfy the reciprocal theorem. Weinstein found the center of twist relative to principal axes passing through the centroid. The present note extends this result to obtain the coordinates of the center of twist in any arbitrarily located transverse coordinate system and also obtains a unique expression for warping of the shaft.

To obtain the warping function, suppose the transverse coordinate axes x, y are parallel to but offset from coordinates $\bar{x}, \bar{y}$ originating at the center of twist; i.e.

$$x = \bar{x}+\eta, \quad y = \bar{y}+\xi, \quad z = \bar{z}, \tag{3}$$

† Any transverse (shear) force on a section gives shear stresses that satisfy equilibrium and boundary conditions. These stresses have a resultant couple that vanishes if the line of action of the shear force passes through the point in the cross-sectional plane that Trefftz called the *center of shear*.

where η, ξ are coordinates of the center of twist in the arbitrarily located coordinate system. For rotation of sections about the center of twist the displacements are

$$u(x,y) = -\alpha z(y-\xi), \quad v(x,y) = \alpha z(x-\eta), \quad w(x,y) = \alpha\psi(x,y). \tag{4}$$

In an elastic shaft the corresponding shear stresses are

$$\tau_{xz} = G\alpha\left(\frac{\partial\psi}{\partial x} - y + \xi\right), \quad \tau_{yz} = G\alpha\left(\frac{\partial\psi}{\partial y} + x - \eta\right). \tag{5}$$

Here $\psi(x,y)$ is the warping shape factor expressed in terms of off-center coordinates x, y. For an arbitrary origin of transverse coordinates, these expressions relate shear stresses to the warping function.

Usually the stresses due to torsion are obtained most easily from Prandtl's stress function $\phi(x,y)$. This alternative to St Venant's method results in a boundary value problem of the Dirichlet type which is simpler to solve than (2). The stresses that satisfy equilibrium and boundary conditions are independent of the origin for transverse coordinate axes (Little, 1973). Consequently, for any particular stress distribution, the warping shape factor $\psi(x,y)$ can be obtained by integration of (5); thus

$$\psi(x,y) = F(x,y) - \xi x + \eta y + \zeta, \tag{6}$$

where F is the warping function other than a displacement due to inclination of z relative to the axis of twist and ζ is an arbitrary constant. Little (1973) provided a proof that for any cross-section the warping is unique. The warping function however depends on the origin for the coordinate system; hence the expression for $\psi(x,y)$ in any off-center coordinate system is related to the function $\bar{\psi}(\bar{x},\bar{y})$ for the center of twist by a coordinate transformation,

$$\psi(x,y) = \bar{\psi}(x-\eta, y-\xi). \tag{7}$$

The center of twist can be defined as the point where the mean square of the axial displacement $w(x,y)$ is a minimum [see Cicala (1935) or Weinstein (1947)]. The location of this point is an extremum of the mean square functional

$$J(\eta,\xi,\zeta) = \iint_A \psi^2(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_A [F(x,y) - \xi x + \eta y + \zeta]^2\,\mathrm{d}x\,\mathrm{d}y,$$

for a cross-section with area A. From $\nabla J(\eta,\xi,\zeta) = \mathbf{0}$ we find that the mean axial displacement vanishes and

$$\begin{bmatrix} I & I_x & -I_y \\ I_x & I_{xx} & -I_{xy} \\ I_y & I_{xy} & -I_{yy} \end{bmatrix} \begin{pmatrix} \zeta \\ \xi \\ \eta \end{pmatrix} = -\begin{pmatrix} \Gamma_0 \\ \Gamma_x \\ \Gamma_y \end{pmatrix}, \tag{8}$$

where

$$I = \iint_A \mathrm{d}x\,\mathrm{d}y, \qquad I_x = \iint_A y\,\mathrm{d}x\,\mathrm{d}y, \qquad I_y = \iint_A x\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_{xy} = \iint_A xy\,\mathrm{d}x\,\mathrm{d}y, \quad I_{xx} = \iint_A y^2\,\mathrm{d}x\,\mathrm{d}y, \quad I_{yy} = \iint_A x^2\,\mathrm{d}x\,\mathrm{d}y,$$

$$\Gamma_0 = \iint_A F\,\mathrm{d}x\,\mathrm{d}y, \quad \Gamma_x = \iint_A yF\,\mathrm{d}x\,\mathrm{d}y, \quad \Gamma_y = \iint_A xF\,\mathrm{d}x\,\mathrm{d}y.$$

For any origin of transverse coordinate axes x, y the location of the center of twist η, ξ and the axial displacement of the origin ζ are evaluated by solving (8). Two special cases are of interest.

(i) *Origin of coordinates at centroid*

If x, y originate at the centroid then $I_x = I_y = 0$; therefore

$$\eta = (I_{xy}\Gamma_y - I_{yy}\Gamma_x)/(I_{xx}I_{yy} - I_{xy}^2),$$
$$\xi = (I_{xx}\Gamma_y - I_{xy}\Gamma_x)/(I_{xx}I_{yy} - I_{xy}^2),$$
$$\zeta = -\Gamma_0/I. \tag{9}$$

Generally the center of twist is not located at the centroid.

(ii) *Origin of coordinates on axis of symmetry*

For cross-sections that are symmetric about the x axis, shear stresses τ_{xz} are anti-symmetric while τ_{yz} are symmetric about the axis of symmetry. Thus $F(x, y)$ is an odd function of y and $\Gamma_0 = \Gamma_y = 0$ as well as $I_x = I_{xy} = 0$; consequently

$$\eta = -\Gamma_x/I_{xx},$$
$$\xi = 0,$$
$$\zeta = 0. \tag{10}$$

This expression is similar to that obtained by Weinstein (1947); however, there is no requirement for the coordinates to originate at the centroid.

It is important to recognize that with an arbitrary origin for transverse coordinates, the warping function can be obtained by integrating (5). Equation (5) differs from that presented in many texts [e.g. Timoshenko and Goodier (1970)] in that it includes the coordinates of the center of twist η, ξ given by (8). This difference is not a simple rigid body translation; it also includes an out-of-plane rotation of the cross-section.

Part of the confusion about the center of twist arises because St Venant's problem has been formulated for infinitesimal twist. This obscures the fact that the center of twist is the only axial line that remains straight in the deformed shaft. The axial line passing through any other point in the cross-section of the undeformed shaft is inclined and deformed by twist; at radius $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}$ from the center of twist, these lines form helices with curvature $\bar{r}\alpha^2/(1 + \bar{r}^2\alpha^2)$ that spiral around the center. This curvature is a higher order term however, that cannot be obtained from the formulation for infinitesimal deformations. Thus this formulation gives the illusion that rotation of the cross-section can be considered to be around any line that is initially parallel to the axis of the cylinder. The curvature of these other lines is not discernible for infinitesimal twist; nevertheless, there is a unique center that can be located from (8). The coordinates of this center of small twist yield a complete and unambiguous expression (6) for warping $\alpha\psi$.

While the center of twist is the only line that remains straight in the twisted shaft, in general there is nonvanishing shear along this line. Hence for cross-sections with no more than one axis of symmetry, the center of twist is not coincident with the centroid or the point $\nabla\phi = \mathbf{0}$ where stresses vanish or the saddle point $\nabla\psi = \mathbf{0}$ of the warping function. All these points are coincident if the cross-section has multiple axes of symmetry.

Example 1: Equilateral triangle cross-section

Torsion in a uniform elastic shaft with equilateral triangular cross-section gives twist α and shear stresses that can be represented by a stress function

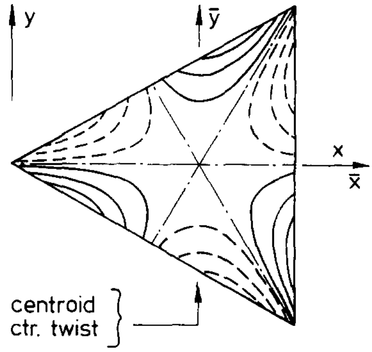

Fig. 1. Contour lines for warping of twisted shaft with equilateral triangle cross-section.

$$\phi = -\frac{Ga^2\alpha}{2}\left(\frac{3y^2}{a^2}-\frac{x^2}{a^2}\right)\left(1-\frac{x}{a}\right),$$

where a is the height of the triangle. This stress function applies if the coordinates originate at a corner of the section as shown in Fig. 1. The shear stresses are given by

$$\tau_{xz} = \partial\phi/\partial y, \quad \tau_{yz} = -\partial\phi/\partial x.$$

To find the warping function at any point we integrate these stresses and substitute into (5); thus

$$\psi = \frac{3x^2y}{2a}-\frac{y^3}{2a}-2xy-\xi x+\eta y.$$

Since x is an axis of symmetry, $F(x,y) = (3x^2y-4axy-y^3)/2a$ is an odd function of y and the location of the center of twist is given by (10); i.e.

$$\eta = 2a/3, \ \xi = 0.$$

This is the centroid of the equilateral triangular cross-section—a section with three axes of symmetry. As a conjecture we propose that the center of twist is coincident with the centroid if the section has more than one axis of symmetry.

Example 2: Circular cross-section with small circular notch

If a grooved shaft has circular cross-section except for a small circular notch, twist α results in shear stresses that can be represented by a stress function

$$\phi = -\frac{G\alpha}{2}\left[\frac{(x^2+y^2-b^2)(x^2+y^2-2ax)}{(x^2+y^2)}\right],$$

where the shaft has radius a, the notch has radius b and the origin of the coordinate system

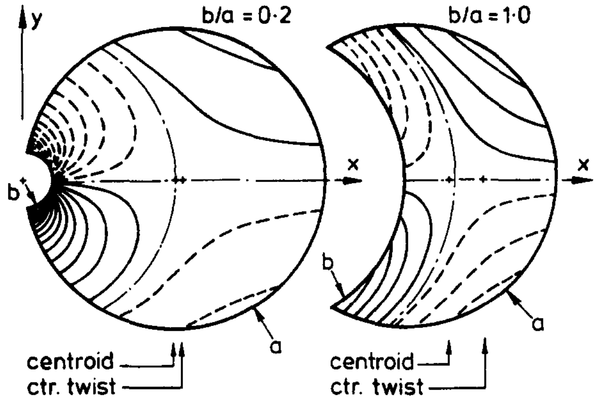

Fig. 2. Contour lines for warping of twisted shaft with notched circular cross-section. For the cross-sections above on the left and right, the relative radii of notch and shaft are $b/a = 0.2$ and 1.0, respectively.

is located at the center of the notch on the periphery of the shaft as shown in Fig. 2. Integration of stresses obtained from this stress function gives the warping function

$$\psi = -\frac{ab^2y}{x^2+y^2} - \xi x + (\eta - a)y + \zeta.$$

This example has an axis of symmetry $y = 0$, so $F(x, y) = -ay - ab^2y(x^2+y^2)^{-1}$ is an odd function of y and $\zeta = 0$. The center of twist is located at $\xi = 0$ and

$$\frac{\eta}{a} = 1 + \frac{3(b/a)^2[4\beta - \sin 4\beta - (b/a)^2(4\beta - 2\sin 2\beta)]}{12\beta + 8\sin 2\beta + \sin 4\beta - 32\cos^5\beta \sin\beta - (b/a)^4(6\beta - 3\sin 2\beta)},$$

where $\beta = \cos^{-1}(b/2a)$. For comparison, the location of the centroid x_c is given by

$$\frac{x_c}{a} = \frac{12\beta + 8\sin 2\beta + \sin 4\beta - 4(b/a)^3 \sin\beta}{6[\beta(2 - b^2/a^2) + \sin 2\beta]}.$$

Shear vanishes at a third point $x_s, 0$ that satisfies $\nabla\phi(x_s, 0) = 0$; this point is a root of

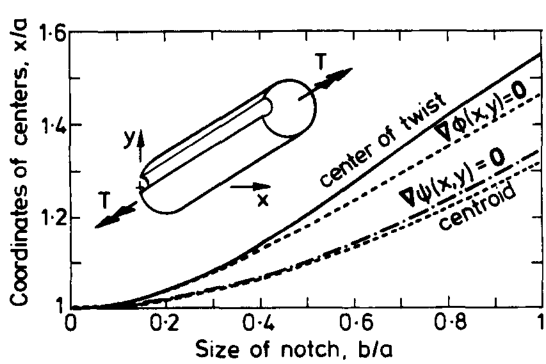

Fig. 3. Coordinates of center of twist, centroid, point where stress vanishes and saddle point of warping function in notched circular shaft as functions of size of the circular notch b/a.

$$(x_s/a)^3-(x_s/a)^2-(b/a)^2=0.$$

Since this cross-section has only one axis of symmetry the center of twist is not coincident with either the centroid or the point where shear vanishes! For this example, the locations of these different points are shown in Fig. 3.

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